Regularization of Orthonormal Vector Sets using Coupled PDE's

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Abstract

We address the problem of restoring, while preserving possible discontinuities, fields of noisy orthonormal vector sets, taking the orthonormal constraints explicitly into account. We develop a variational solution for the general case where each image feature may correspond to multiple n-D orthogonal vectors of unit norms. We first formulate the problem in a new variational framework, where discontinuities and orthonormal constraints are preserved by means of constrained minimization and Φ-functions regularization, leading to a set of coupled anisotropic diffusion PDE's. A geometric interpretation of the resulting equations, coming from the field of solid mechanics, is proposed for the 3D case. Two interesting restrictions of our framework are also tackled : the regularization of 3D rotation matrices and the Direction diffusion (the parallel with previous works is made). Finally, we present a number of denoising results and applications.

1. Introduction

For many years, data regularization with discontinuities preservation has been heavily studied in the computer vision community. The variational framework, based on functional minimizations via diffusion PDE's evolutions has proved its efficiency for scalar data regularization (in particular, within the Φ-functions theory). We can cite for instance, Alvarez *et al.* [1, 2], Aubert *et al.* [10], Chambolle & Lions [7], Chan [5], Cohen [11], Kornprobst & Deriche [15, 16, 17], Malladi & Sethian [18], Mumford & Shah [20, 31], Morel [19], Nordström [21], Osher & Rudin [26], Perona & Malik [23], Proesman *et al.* [25], Sapiro [6, 28, 29, 30], Weickert [38, 39], You [41], . . .

More recently, *vector field regularization* with vector diffusion PDE's, while preserving discontinuities, has become an active research area, due to the large number of possible applications, including various computer vision tasks : multivalued image restoration (in particular color images) [5, 14, 30, 32, 36, 37, 38], regularization of optical flows and direction fields [8, 22, 33], image inpainting [3, 9], scale space analysis [2, 40], . . . The introduction of minimization on *constrained manifolds* has also permitted to regularize more complex and specific data, using the knowledge of the solution space. We can mention for instance the case of vector directions and chromaticity diffusion, based on the harmonic map framework [33], on total variation minimization [8], and the diffusion of vector data on arbitrary manifold [4]. Constrained minimizations usually lead to sets of coupled PDE's where the coupling between data is taken into account.

The aim of this paper is to propose a variational framework, allowing to regularize an original and interesting type of constrained vector data : images of n-*D orthonormal vector sets*. Many informations can be represented (even partially) by such images, since data may be decomposed into orthogonal vectors : We can cite for instance images or sequences of rotation matrices, diffusion tensors after eigenvalue decomposition, . . . The idea is then to find a process that diffuse directly such structures, avoiding transformations with eventual loss of informations (the case of 3D rotation decompositions will be discussed). It also better considers the correlation between the data, while ensuring that the resulting vector structure stays orthonormal.

In section 2, we formalise mathematically the problem, then propose a PDE based solution, coming from a constrained gradient descent of a Φ-function minimization (section 3). Then, we are interested more precisely in the simple case where each image feature is a 3D orthonormal basis, and propose a physical interpretation coming from the field of solid mechanics (section 4) as well as apply our algorithm to regularize fields of 3D rotation matrices. Previous works on vector direction diffusion (as described in [8, 33]) can also be integrated into our proposed framework (section 5), as a simple particular case : The regularization of *sets of single vectors* under orthonormal constraints. Finally, we present possible applications and some regularization results of 2D and 3D orthonormal basis fields (section 6), including restoration of video camera motion and chromaticity noise in color images.

2. Notations and context

We consider m vector images $\mathbf{I}^{[k]} : \Omega \rightarrow \mathbb{R}^n$, supposed twice differentiable and defined on a closed set Ω of $\mathbb{R}^p$ (where $1 \leq k \leq m \leq n$, and usually $p = 1, 2, 3$).
$I_i^{[k]} : \Omega \rightarrow \mathbb{R}$ denotes the scalar image corresponding to the i^{th} vector component of $\mathbf{I}^{[k]}$: $\quad \forall M \in \Omega,$

$$\mathbf{I}^{[k]}(M) = \left(\; I_1^{[k]}(M) \; , \; I_2^{[k]}(M) \; , \; \ldots \; , \; I_n^{[k]}(M) \; \right)$$

We define $\boldsymbol{\mathcal{B}}$, the set of the m vector images $\mathbf{I}^{[k]}$:

$$\forall M \in \Omega \; , \quad \boldsymbol{\mathcal{B}}(M) = \left\{ \; \mathbf{I}^{[1]}(M) \; , \; \mathbf{I}^{[2]}(M) \; , \ldots , \; \mathbf{I}^{[m]}(M) \; \right\}$$

We also suppose the following *orthonormal* constraints :

$$\forall M \in \Omega, \qquad \mathbf{I}^{[k]}(M) \, . \, \mathbf{I}^{[l]}(M) = \delta_{kl} = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases} \tag{1}$$

where $\;\mathbf{I}(M) \, . \, \mathbf{J}(M) = \sum_{i=1}^{n} I_i(M) \; J_i(M)\;$ is the usual dot product of two vectors. It just means that :

$\forall M \in \Omega \; , \quad \boldsymbol{\mathcal{B}}(M)$ is *a set of m orthonormal n-D vectors*.

In this paper, we propose a way to regularize any data that can be represented as an image $\boldsymbol{\mathcal{B}}$ of orthonormal vector sets, using anisotropic diffusion PDE's.

An interesting case is reached when $m = n$: $\boldsymbol{\mathcal{B}}(M)$ is then *an orthonormal vector basis*. Let us illustrate the particular example of 3D rotation matrix fields $\boldsymbol{\mathcal{R}}$. In this case, each pixel $\boldsymbol{\mathcal{R}}(M)$ is a 3×3 rotation matrix ($m = n = 3$)

$$\boldsymbol{\mathcal{R}}(M) = \begin{pmatrix} I_1(M) & J_1(M) & K_1(M) \\ I_2(M) & J_2(M) & K_2(M) \\ I_3(M) & J_3(M) & K_3(M) \end{pmatrix}$$

that can be seen as a *3D direct orthonormal vector basis $\boldsymbol{\mathcal{B}}$*, since a rotation matrix is orthogonal.

$$\forall M \in \Omega \quad \boldsymbol{\mathcal{B}}(M) = \{ \; \mathbf{I}(M) \; , \; \mathbf{J}(M) \; , \; \mathbf{K}(M) \; \}$$

verifying the orthonormal constraints :

$$\begin{cases} \mathbf{I}(M) \perp \mathbf{J}(M) \; , \quad \mathbf{I}(M) \perp \mathbf{K}(M) \; , \quad \mathbf{J}(M) \perp \mathbf{K}(M) \\ \|\mathbf{I}(M)\| = \|\mathbf{J}(M)\| = \|\mathbf{K}(M)\| = 1 \end{cases}$$

($\|\mathbf{I}(M)\| = \sqrt{\mathbf{I}(M) \, . \, \mathbf{I}(M)}$ is the usual $\mathcal{L}_2$ vector norm).

Then, one possible application of our proposed algorithm is the regularization of 3D rotation matrix images. For instance, such images $\boldsymbol{\mathcal{B}}$ are estimated from real video sequences, using camera motion estimation techniques (see section 4.3), or obtained from an eigenvalue decomposition of 3D matrices (diffusion tensor imaging,. . .) (Fig.1).

More generally, our method will restore n-D orthonormal vectors sets $\boldsymbol{\mathcal{B}}$ which can represent a lot of different data types. If those images are computed from real data, they may be noisy and a regularization process is then useful.

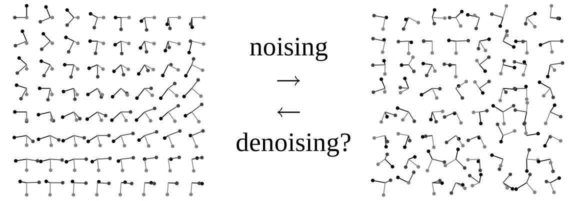


Figure 1. Restoring an orthonormal field

3. A variational formulation

Our goal is to find a regularized version $\boldsymbol{\mathcal{B}}$ of an initial image $\boldsymbol{\mathcal{B}}_0$ of *orthonormal vector sets* :

$$\boldsymbol{\mathcal{B}}_0(M) = \left\{ \; \mathbf{I}^{[1]}(M)_0 \; , \; \mathbf{I}^{[2]}(M)_0 \; , \; \ldots \; , \; \mathbf{I}^{[m]}(M)_0 \; \right\}$$

preserving the orthonormal structure of the vector sets.

3.1. Unconstrained vector regularization

We propose to find $\boldsymbol{\mathcal{B}}$ as the solution of an energy minimization, following the well know idea of Φ-function diffusion, used to restore scalar images (see for instance [10, 16, 17, 23]), and more recently, vector fields [5, 14, 30, 32, 36, 37, 38]). We quickly remind the idea. A vector image $\mathbf{I}$ can be anisotropically smoothed (denoising with preservation of discontinuities), by minimizing :

$$E_{\text{diff}}(\mathbf{I}) = \int_\Omega \left[\frac{\alpha}{2} \; \|\mathbf{I} - \mathbf{I}_0\|^2 + \Phi(\|\nabla \mathbf{I}\|) \right] \; d\Omega \tag{2}$$

where $\|\nabla \mathbf{I}\| = \sqrt{\sum_{i=1}^{n} \|\nabla I_i\|^2}$ is defined as the *vector gradient norm* which measures a global vector variation (norm and orientation). The fixed parameter $\alpha \in \mathbb{R}$ prevents the final solution from being too different from the initial image. The function $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ is a *diffusion function*, which controls the regularization behaviour. A lot of different Φ-functions have already been proposed in the literature related to scalar image restoration : Minimal surfaces [10], Geman & McClure [12], Perona & Malik [23], Total variation [27], Tikhonov [35], . . . (choosing the right Φ-function depends on the application).
One way of minimizing the functional $E_{\text{diff}}(\mathbf{I})$, is to calculate the corresponding *vector Lagrangian* $\boldsymbol{\mathcal{L}}(E_{\text{diff}}(\mathbf{I})) \in \mathbb{R}^n$ which is, using a component by component writing style :

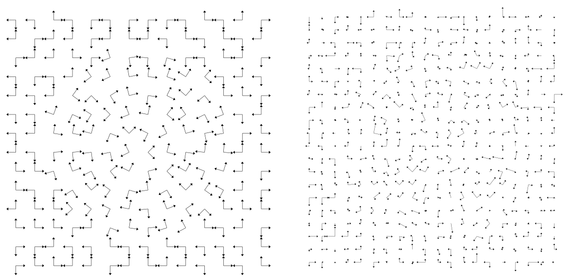
$$\mathcal{L}(E_{\text{diff}}(\mathbf{I}))_i = \alpha \; (I_i - I_{i_0}) - \text{div} \left(\frac{\Phi'(\|\nabla \mathbf{I}\|)}{\|\nabla \mathbf{I}\|} \nabla I_i \right)$$

Then use a vector gradient descent : $\frac{\partial \mathbf{I}}{\partial t} = -\boldsymbol{\mathcal{L}}(E_{\text{diff}}(\mathbf{I}))$:

$$\begin{cases} \mathbf{I}_{(t=0)} = \mathbf{I}_0 \\ \frac{\partial I_i}{\partial t} = \alpha \; (I_{i_0} - I_i) + \text{div} \left(\frac{\Phi'(\|\nabla \mathbf{I}\|)}{\|\nabla \mathbf{I}\|} \nabla I_i \right) \end{cases} \tag{3}$$

until convergence. ($i \in [1 \ldots n]$, there are n scalar PDE's)

For our purpose of orthonormal vector set regularization, a naive idea would be to use such diffusion PDE's (3) on each vector $\mathbf{I}_0^{[k]}$ of the orthonormal vector set $\boldsymbol{\mathcal{B}}_0$, then reconstruct the final vector set image $\boldsymbol{\mathcal{B}}$ with the resulting smoothed vectors. Fig.2 shows such a result, using the Tikhonov function on a 2D synthetic image of orthonormal bases (mixture of direct and indirect bases).



a) Original field $\boldsymbol{\mathcal{B}}_0$ b) "Regularized" field $\boldsymbol{\mathcal{B}}$

Figure 2. decoupled diffusion of 2D bases.

Unfortunately, this unconstrained method breaks the orthonormal properties : vector norms and orthogonal angles are not intrinsically preserved. We must explicitly introduce *orthonormal constraints*, in the minimization process. Interesting work on constrained minimization of vector fields has already been done in [4, 8, 33, 37]. The idea was to regularize a normalised vector field, preserving the unitary vector norm. It yielded a set of PDE's taking the coupling between vector components into account. Anyway, no approach dealing with multiple vectors were proposed (which is needed in our case). Actually, these works can be seen as a part of our proposed framework (section 5).

3.2. Adding orthonormal constraints

In order to regularize the orthonormal vector set $\boldsymbol{\mathcal{B}}(M)_0$ while preserving the orthonormal properties, we propose a constrained minimization of the following functional :

$$E(\boldsymbol{\mathcal{B}}) = \int_\Omega \sum_{k=1}^{m} \left[\frac{\alpha}{2} \, \|\mathbf{I}^{[k]} - \mathbf{I}_0^{[k]}\|^2 + \Phi(\|\nabla \mathbf{I}^{[k]}\|) \right] \, d\Omega \tag{4}$$

with respect to the m vector functions $\mathbf{I}^{[k]}$, subject to the orthonormal constraints :

$$\forall M \in \Omega, \qquad \mathbf{I}^{[p]}(M) \,.\, \mathbf{I}^{[q]}(M) = \delta_{pq} = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{if } p \neq q \end{cases}$$

We denote by $\boldsymbol{\mathcal{L}}(E)^{[k]} \in \mathbb{R}^n$, the *Lagrangian vector* of the energy $E(\boldsymbol{\mathcal{B}})$ subject to the vector $\mathbf{I}^{[k]}$:

$$\mathcal{L}(E)_i^{[k]} = \alpha \, (I_i^{[k]} - I_{i_0}^{[k]}) - \text{div} \left(\frac{\Phi'(\|\nabla \mathbf{I}^{[k]}\|)}{\|\nabla \mathbf{I}^{[k]}\|} \nabla I_i^{[k]} \right)$$

It is obviously the same expression than $\boldsymbol{\mathcal{L}}(E_{\text{diff}}(\mathbf{I}))$ defined in (3), for the unconstrained case. The constraints are then introduced by adding m^2 Lagrange multipliers $\lambda_{pq} : \Omega \to \mathbb{R}$ (where $p, q \in [1 \ldots m]$) to the functional $E(\boldsymbol{\mathcal{B}})$, where each λ_{pq} is associated with the constraint :

$$\forall M \in \Omega, \quad \mathbf{I}^{[p]} \,.\, \mathbf{I}^{[q]} = \delta_{pq}$$

It leads to the *unconstrained minimization* of the following functional, with respect to $\mathbf{I}^{[k]}$ and λ_{pq} :

$$E^*(\boldsymbol{\mathcal{B}}, \lambda) = E(\boldsymbol{\mathcal{B}}) + \int_\Omega \sum_{(p,q) \in [1 \ldots m]} \lambda_{pq} \, (\mathbf{I}^{[p]} \,.\, \mathbf{I}^{[q]} - \delta_{pq}) \, d\Omega$$

In fact, as the dot product and δ_{pq} are symmetric, the constraints $\mathbf{I}^{[p]} \,.\, \mathbf{I}^{[q]} = \delta_{pq}$ and $\mathbf{I}^{[q]} \,.\, \mathbf{I}^{[p]} = \delta_{qp}$ are the same, and should need only one Lagrange multiplier. The two Lagrange multipliers λ_{pq} and λ_{qp} are then equal.
When the constrained minimum is reached, the Euler-Lagrange equations corresponding to $E^*(\boldsymbol{\mathcal{B}}, \lambda)$ are then :

$$\begin{cases} \boldsymbol{\mathcal{L}}(E)^{[k]} + 2 \sum_{l=1}^{m} \lambda_{lk} \, \mathbf{I}^{[l]} = 0 \\ \mathbf{I}^{[p]} \,.\, \mathbf{I}^{[q]} = \delta_{pq} \qquad\qquad (k, p, q \in [1 \ldots m]). \end{cases} \tag{5}$$

Finding the λ_{kl} reached at the minimum is performed as follow : we take the dot product of the first m equations (first row) with each vector $\mathbf{I}^{[p]}$ $(p \in [1 \ldots m])$, then simplify it using the relations $\mathbf{I}^{[p]} \,.\, \mathbf{I}^{[l]} = \delta_{pl}$ (second row) :

$$\forall k, l \in [1 \ldots m], \quad \lambda_{kl} = -\frac{\boldsymbol{\mathcal{L}}(E)^{[k]} \,.\, \mathbf{I}^{[l]}}{2}$$

Finally, replacing all the λ_{lk} in the Euler-Lagrange system eq.(5) gives the gradient descent *preserving the orthonormal constraints* :

$$\frac{\partial \mathbf{I}^{[k]}}{\partial t} = \sum_{l=1}^{m} \left(\boldsymbol{\mathcal{L}}(E)^{[l]} \,.\, \mathbf{I}^{[k]} \right) \mathbf{I}^{[l]} - \boldsymbol{\mathcal{L}}(E)^{[k]} \tag{6}$$

where

$$\mathcal{L}(E)_i^{[k]} = \alpha \, (I_i^{[k]} - I_{i_0}^{[k]}) - \text{div} \left(\frac{\Phi'(\|\nabla \mathbf{I}^{[k]}\|)}{\|\nabla \mathbf{I}^{[k]}\|} \nabla I_i^{[k]} \right)$$

is the vector Lagrangian of the corresponding *unconstrained functional*. It can be seen as a *pure diffusion vector force* (Section 4.2 provides a simple physical interpretation of this vector).

The proposed equation (6) is a set of m coupled vector PDE's (where the coupling between vectors *and* vector components is clearly present), which allows to regularize any field of orthonormal vector sets, preserving the orthonormal structure of the vectors during the PDE evolution. It is also worth to mention that it naturally preserves the direct or indirect feature of the bases, since the evolution is smooth (rotation matrices as described in section 2 cannot then transform to rotoinversions).

4. The case of 3D orthonormal basis

4.1. Notations and equations

We are now interested in the constrained diffusion of 3D orthonormal basis fields ($m = n = 3$), using eq.(6). For simplicity reasons, we denote the three basis vectors by :

$$\mathbf{I} = \mathbf{I}^{[1]}\ ,\quad \mathbf{J} = \mathbf{I}^{[2]}\quad \text{and}\quad \mathbf{K} = \mathbf{I}^{[3]}\quad \text{then}\quad \mathcal{B} = \{\ \mathbf{I}, \mathbf{J}, \mathbf{K}\ \}$$

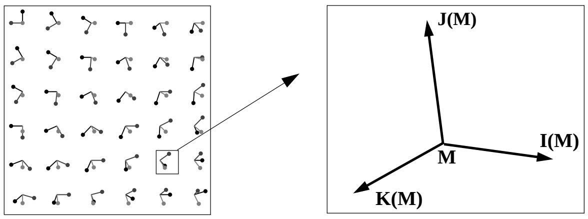


Figure 3. Example of a 3D orthonormal field.

In order to regularize $\mathcal{B}$ while preserving discontinuities, we minimize the functional (4), with $m = n = 3$:

$$E(\mathcal{B}) = \int_\Omega \begin{array}{l} \frac{\alpha}{2}\ (\|\mathbf{I} - \mathbf{I}_0\|^2 + \|\mathbf{J} - \mathbf{J}_0\|^2 + \|\mathbf{K} - \mathbf{K}_0\|^2\ ) \\ +\Phi(\|\nabla \mathbf{I}\|) + \Phi(\|\nabla \mathbf{J}\|) + \Phi(\|\nabla \mathbf{K}\|)\ \ d\Omega \end{array}$$

Using eq.(6), we find the corresponding constrained set of 3D vector diffusion PDE's :

$$\left\{ \begin{array}{l} \mathbf{I}_t\ = \mathbf{f}^{\mathbf{I}}\ -\ (\mathbf{f}^{\mathbf{I}}.\mathbf{I})\ \mathbf{I} - (\mathbf{f}^{\mathbf{J}}.\mathbf{I})\ \mathbf{J} - (\mathbf{f}^{\mathbf{K}}.\mathbf{I})\ \mathbf{K} \\ \mathbf{J}_t\ = \mathbf{f}^{\mathbf{J}}\ -\ (\mathbf{f}^{\mathbf{I}}.\mathbf{J})\ \mathbf{I} - (\mathbf{f}^{\mathbf{J}}.\mathbf{J})\ \mathbf{J} - (\mathbf{f}^{\mathbf{K}}.\mathbf{J})\ \mathbf{K} \\ \mathbf{K}_t = \mathbf{f}^{\mathbf{K}} - (\mathbf{f}^{\mathbf{I}}.\mathbf{K})\ \mathbf{I} - (\mathbf{f}^{\mathbf{J}}.\mathbf{K})\ \mathbf{J} - (\mathbf{f}^{\mathbf{K}}.\mathbf{K})\ \mathbf{K} \end{array} \right. \qquad (7)$$

where $\mathbf{f}^{\mathbf{u}}$ is the *unconstrained diffusion vector* defined by :

$$f_i^{\mathbf{u}} = \alpha\ (u_{i_0} - u_i) + \text{div}\left(\frac{\Phi'(\|\nabla \mathbf{u}\|)}{\|\nabla \mathbf{u}\|} \nabla u_i \right) \qquad (i = 1, 2, 3)$$

4.2. A physical interpretation

$\mathcal{B}(M) = \{\ \mathbf{I}(M), \mathbf{J}(M), \mathbf{K}(M)\ \}$ can be seen as a solid object composed of three orthogonal rigid stems of unit length, fixed at the same point M, and submitted to forces $\mathbf{f}^{\mathbf{I}}$, $\mathbf{f}^{\mathbf{J}}$ and $\mathbf{f}^{\mathbf{K}}$ respectively (Fig.4).

A rotation around M is obviously the only motion that can perform $\mathcal{B}$. Actually, each force $\mathbf{f}^{\mathbf{I}}$, $\mathbf{f}^{\mathbf{J}}$ and $\mathbf{f}^{\mathbf{K}}$ induces a mechanic momentum on this object :

$$\Omega_{\mathbf{I}} = \mathbf{I} \times \mathbf{f}^{\mathbf{I}}\ ,\quad \Omega_{\mathbf{J}} = \mathbf{J} \times \mathbf{f}^{\mathbf{J}}\ ,\ \text{and}\ \ \Omega_{\mathbf{K}} = \mathbf{K} \times \mathbf{f}^{\mathbf{K}}$$

Where $\times$ designates the usual cross product in $\mathbb{R}^3$. Then, the total momentum applied to the object $\mathcal{B}$ is given by :

$$\Omega = \Omega_{\mathbf{I}} + \Omega_{\mathbf{J}} + \Omega_{\mathbf{K}} = \mathbf{I} \times \mathbf{f}^{\mathbf{I}} + \mathbf{J} \times \mathbf{f}^{\mathbf{J}} + \mathbf{K} \times \mathbf{f}^{\mathbf{K}}$$

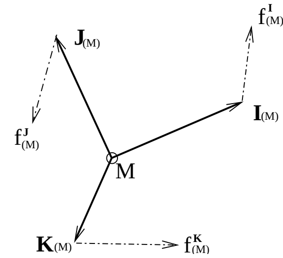


Figure 4. A solid object $\mathcal{B}$,submitted to forces.

If we suppose that $\mathcal{B}$ has an unit moment of inertia, we can express the velocities $\mathbf{v}^{\mathbf{I}}$, $\mathbf{v}^{\mathbf{J}}$ and $\mathbf{v}^{\mathbf{K}}$ at each free extremity of the stems, corresponding to *the constrained movement* of the solid :

$$\left\{ \begin{array}{l} \mathbf{v}^{\mathbf{I}} = \Omega \times \mathbf{I} \\ \mathbf{v}^{\mathbf{J}} = \Omega \times \mathbf{J} \\ \mathbf{v}^{\mathbf{K}} = \Omega \times \mathbf{K} \end{array} \right. \quad \text{with}\quad \Omega = \mathbf{I} \times \mathbf{f}^{\mathbf{I}} + \mathbf{J} \times \mathbf{f}^{\mathbf{J}} + \mathbf{K} \times \mathbf{f}^{\mathbf{K}}$$

Developing these expressions, using the double vector product formula $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u}.\mathbf{w})\ \mathbf{v} - (\mathbf{u}.\mathbf{v})\ \mathbf{w}$ and the orthogonal properties $\mathbf{I}^{[k]}\ .\ \mathbf{I}^{[l]} = \delta_{kl}$ leads to :

$$\left\{ \begin{array}{l} \mathbf{v}^{\mathbf{I}}\ = \mathbf{f}^{\mathbf{I}}\ -\ (\mathbf{f}^{\mathbf{I}}.\mathbf{I})\ \mathbf{I} - (\mathbf{f}^{\mathbf{J}}.\mathbf{I})\ \mathbf{J} - (\mathbf{f}^{\mathbf{K}}.\mathbf{I})\ \mathbf{K} \\ \mathbf{v}^{\mathbf{J}} = \mathbf{f}^{\mathbf{J}}\ -\ (\mathbf{f}^{\mathbf{I}}.\mathbf{J})\ \mathbf{I} - (\mathbf{f}^{\mathbf{J}}.\mathbf{J})\ \mathbf{J} - (\mathbf{f}^{\mathbf{K}}.\mathbf{J})\ \mathbf{K} \\ \mathbf{v}^{\mathbf{K}} = \mathbf{f}^{\mathbf{K}} - (\mathbf{f}^{\mathbf{I}}.\mathbf{K})\ \mathbf{I} - (\mathbf{f}^{\mathbf{J}}.\mathbf{K})\ \mathbf{J} - (\mathbf{f}^{\mathbf{K}}.\mathbf{K})\ \mathbf{K} \end{array} \right.$$

A velocity is an infinitesimal variation of a vector in the time :

$$\frac{\partial \mathbf{I}}{\partial t} = \mathbf{v}^{\mathbf{I}}\ ,\quad \frac{\partial \mathbf{J}}{\partial t} = \mathbf{v}^{\mathbf{J}}\ ,\quad \frac{\partial \mathbf{K}}{\partial t} = \mathbf{v}^{\mathbf{K}}$$

And, by choosing the following forces applied to the stems :

$$f_i^{\mathbf{u}} = \alpha\ (u_{i_0} - u_i) + \text{div}\left(\frac{\Phi(\|\nabla \mathbf{u}\|)}{\|\nabla \mathbf{u}\|} \nabla u_i \right)$$

We find the expected orthonormal constrained regularization PDE's eq.(7), coming from the variational formulation. The interpretation is then simple : Constrained minimization of the functional (4) is equivalent to perform a movement of a rigid and fixed object $\mathcal{B}$ submitted to three *diffusion forces* $\mathbf{f}^{\mathbf{u}}$ which tend to rotate the object in order to minimize the mechanic energy of the system (a kind of vector basis gradient).

4.3. Application to 3D rotation matrices

If one wants to regularize rotation matrix fields, a natural idea is to decompose the matrices into more simple data that are easy to regularize (usually Euler angles, unit quaternions or rotation vectors), then reconstruct the final rotation field from these smoothed data (Fig.5).

Anyway this method has some drawbacks : The conversions induce numerical errors and are not unique. It introduces

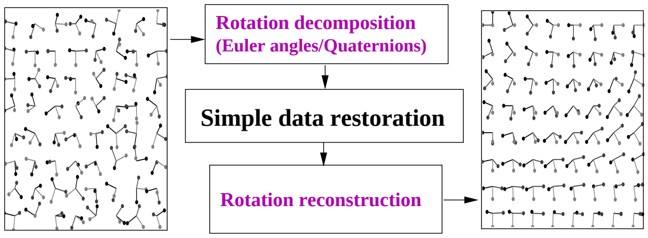


Figure 5. A natural (but not effective) idea, for 3D rotations regularization.

annoying discontinuities in the decomposed data, even if the initial rotation field is continuous.
These discontinuities are coming from :

- The 2π-periodicity ambiguity of the Euler angles or the rotation vector norms.
- The double representation of a single rotation by two equivalent quaternions q and $-q$.

It has a large influence on the regularization process by detecting non-existent discontinuities and modifying the diffusion behaviour.

Actually, our proposed framework (section 3) deals easily with this problem : An image of 3D rotation matrices can be represented as a field of 3D orthonormal direct bases (as described in section 2), without any conversion and loss of informations. Then, the orthonormal constrained PDE's eq.(6) allows to regularize the rotation field. The method is direct and effective (coupling between vectors are explicitly taken into account).

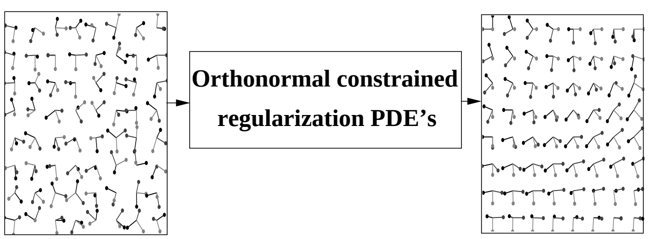


Figure 6. A more simple method.

Rotation matrix regularization can be a part of restoring a video camera sequence (Fig.7) : Taking a real movie sequence as an input, a first process estimates the camera motion, then outputs two sequences, one corresponding to the camera translation (change of the view point) and the second to the camera rotation (change of the view angle). These outputs may be noisy (the motion estimation algorithm often uses correspondence points which are very sensitive to the noise). The rotation sequence is easily restored using our orthonormal constrained equation (6), while the translation part is regularized using a classic vector regularization approach (as in [5, 14, 30, 32, 36, 37, 38]).

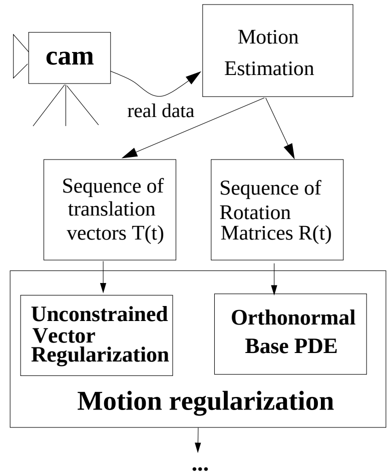


Figure 7. Camera motion regularization

For instance, this process allows smoother reprojections of virtual 3D objects on the original movie. We show one result of this method in section 6.

5. A parallel with direction diffusion

Vector direction diffusion of vectors has already been studied in [4, 8, 33, 37]. Actually, this problem can be seen as a particular case of our orthonormal vector set framework, where the vector sets $\boldsymbol{\mathcal{B}}(M)$ are restricted to a single vector $\boldsymbol{\mathcal{B}}(M) = \{\mathbf{I}(M)\}$.
Indeed, the orthonormal constraints eq.(1) are reduced to the unitary norm constraint : $\forall M \in \Omega, \quad \|\mathbf{I}(M)\| = 1$.
The corresponding functional eq.(4) also reduces to :

$$E(\mathbf{I}) = \int_\Omega \left[\alpha\ \|\mathbf{I} - \mathbf{I}_0\|^2 + \Phi(\|\nabla \mathbf{I}\|)\right]\ d\Omega$$

and the resulting constrained diffusion PDE is (eq.(6)) :

$$\frac{\partial \mathbf{I}}{\partial t} = (\boldsymbol{\mathcal{L}}(E)\ .\ \mathbf{I})\ \mathbf{I} - \ \boldsymbol{\mathcal{L}}(E) \tag{8}$$

where $\mathcal{L}(E)_i = \alpha\ (I_i - I_{i_0}) - \operatorname{div}\left(\frac{\Phi'(\|\nabla \mathbf{I}\|)}{\|\nabla \mathbf{I}\|} \nabla I_i\right)$.

This equation can be highly simplified : From the spatial derivation of $\|\mathbf{I}(M)\|^2 = 1$, we find :

$$\mathbf{I}\ .\ \mathbf{I}_x = \mathbf{I}\ .\ \mathbf{I}_y = 0 \ \text{ and } \ \Delta \mathbf{I}\ .\ \mathbf{I} = -\|\nabla \mathbf{I}\|^2 \tag{9}$$

Developing the divergence for each $\mathcal{L}(E)_i$:

$$\operatorname{div}(A) = \frac{\Phi'(\|\nabla \mathbf{I}\|)}{\|\nabla \mathbf{I}\|}\ \Delta \mathbf{I}_i + \nabla\left(\frac{\Phi'(\|\nabla \mathbf{I}\|)}{\|\nabla \mathbf{I}\|}\right)\ .\ \nabla I_i$$

where $A = \frac{\Phi^{'}(\|\nabla \mathbf{I}\|)}{\|\nabla \mathbf{I}\|} \nabla I_i$.

If we note $[\ a\ ,\ b\ ]^T = \nabla \left(\frac{\Phi^{'}(\|\nabla \mathbf{I}\|)}{\|\nabla \mathbf{I}\|} \right)$, then :

$$\text{div}\,(A)\ \ .\ \mathbf{I} = \frac{\Phi^{'}(\|\nabla \mathbf{I}\|)}{\|\nabla \mathbf{I}\|}\ (\Delta \mathbf{I}\ .\ \mathbf{I}) + a\ (\mathbf{I}_x\ .\ \mathbf{I}) + b\ (\mathbf{I}_y\ .\ \mathbf{I})$$

Using eq.(9), we get : $\text{div}\,(A)\ \ .\ \mathbf{I} = -\Phi^{'}(\|\nabla \mathbf{I}\|)\ \|\nabla \mathbf{I}\|$

Then, the proposed diffusion PDE eq.(8) becomes :

$$\frac{\partial I_i}{\partial t} = \left\{ \begin{array}{l} div \left(\frac{\Phi^{'}(\|\nabla \mathbf{I}\|)}{\|\nabla \mathbf{I}\|}\ \nabla I_i \right) + \Phi^{'}(\|\nabla \mathbf{I}\|)\ \|\nabla \mathbf{I}\|\ I_i \\ +\alpha\ (I_{i_0} - (\mathbf{I}_0\ .\ \mathbf{I})\ I_i) \end{array} \right. \tag{10}$$

This vector diffusion PDE can be used to regularize vector direction fields, using various Φ-functions. Note that the PDE's proposed in [8, 33] are a restriction of the eq.(10) to $\alpha = 0$ and $\Phi(s) = s^r$ $(r = 1, 2)$. The beautiful thing of our *orthonormal vector sets regularization* method, is the implicit unification of previous works on normalized vectors diffusion, into a larger framework. It naturally extends norm constrained regularizations to arbitrary Φ-functions, as well as it shows the corresponding physical interpretation.

6. Results

We illustrate the applications of the equations (6),(7),(10) proposed in this paper with some examples :

- Fig.8 and Fig.9,d,e,f show applications of our orthonormal constrained PDE's on synthetic 2D and 3D basis fields. Note how the constraints are necessary to get an acceptable result.
- Fig.9,a,b,c illustrates an useful application : the regularization of real camera orientation sequence, using eq.(7), as described in section 4.3. The corresponding Euler angles of the sequences are displayed (but were not used for the regularization !). Virtual 3D objects reprojected on this movie, using the restored motion sequence have then a smoother motion (sequence provided by REALVIZ : *http://www.realviz.com*).
- We can also use our orthonormal PDE's eq.(6) reduced to eq.(10), in order to smooth vector direction fields, while preserving discontinuities (Fig.9,g,h,i). This example use the hypersurface Φ-function : $\Phi = 2(\sqrt{1+u^2} - 1)$.
- This last equation allows also to restore chromaticity data in color images (Fig.9,j,k,l), as mentioned in [8, 33, 37] : The chromaticity information is given by the orientation of the color vector, while the brightness is its norm. The knowledge of a chromaticity noise allows a better restoration than with unconstrained diffusion PDE's.

Conclusions & Perspectives

In this paper, we addressed the problem of restoring, while preserving possible discontinuities, fields of noisy orthonormal vector sets. We have formulated the problem in a variational framework based on the Φ-function principle, where discontinuities and orthonormal basis constraints are preserved thanks to coupled anisotropic diffusion PDE's. The importance of the basis constraint was clearly shown, as well as the physical interpretation of the proposed equations, in the simple 3D case. We also made the link with the previous work on norm constrained evolution of vector fields. We applied these constrained diffusion equations in order to restore 2D and 3D orthonormal basis fields, and shown an original application : The regularization of a video camera motion. As a perspective, we are in the process to use the corresponding PDE's, in order to restore *diffusion tensor MRI images*.

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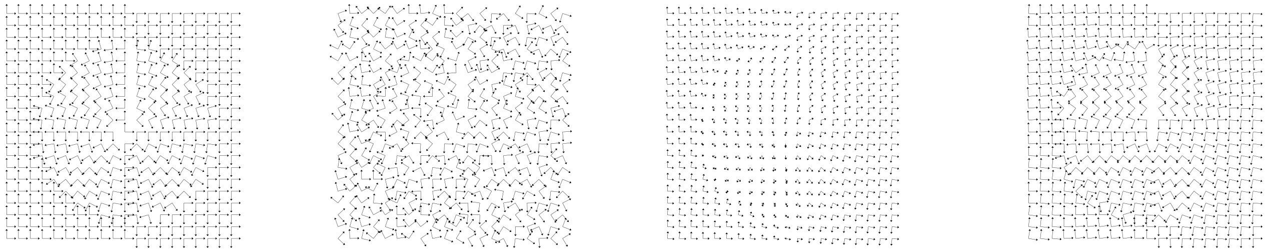
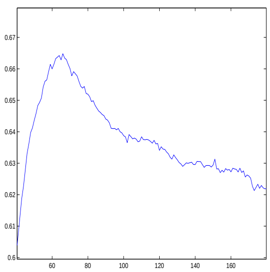
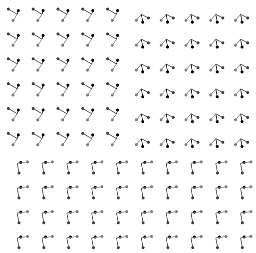


Synthetic field of 2D bases. With noisy orientations ($\sigma = 70^o$). With unconstrained PDE's. With our orthonormal constrained PDE's.

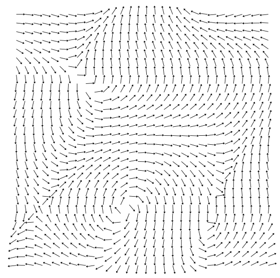
Figure 8. Importance of the orthonormal constraints for the regularization process.



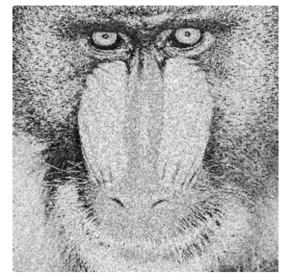
a) One Euler angle of a *real* camera sequence.



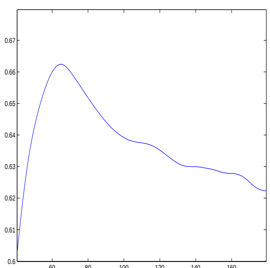
d) Synthetic 3D rotation field.



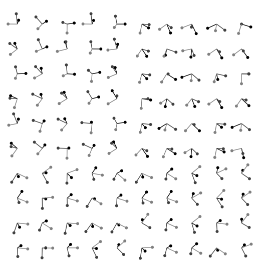
g) Vector direction field.



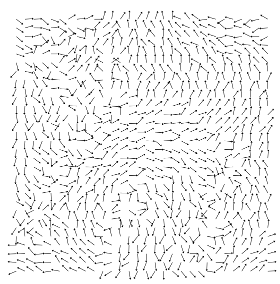
j) Noisy chromaticity color image.



b) Rotation regularization with eq(7) (after 10 it.).

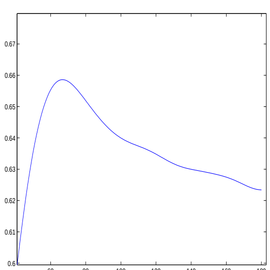


e) Noisy 3D rotation field.

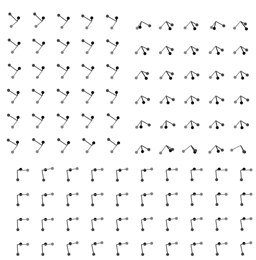


h) Noisy direction field.

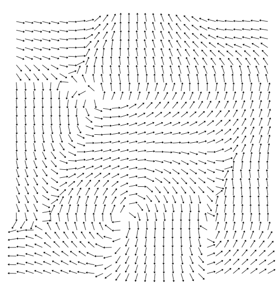
k) Unconstrained color restoration.



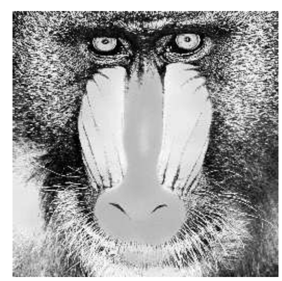
c) Rotation regularization with eq(7) (after 50 it.).



f) Restored field (eq.(7)).



i) Restored direction field (eq.(10)).



l) Constrained restoration (eq.(10)).

Figure 9. Some possible applications of our orthonormal vector set framework : camera orientation regularization, rotation field restoration, direction diffusion, color image restoration, . . . (arranged column by column)